

# Coupled identical localized fermionic chains with quasi-random disorder

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We analyze the ground state localization properties of an array of identical interacting spinless fermionic chains with quasi-random disorder, using non-perturbative Renormalization Group methods. In the single or two chains case localization persists while for a larger number of chains a different qualitative behavior is generically expected, unless the many body interaction is vanishing. This is due to number theoretical properties of the frequency, similar to the ones assumed in KAM theory, and cancellations due to Pauli principle which in the single or two chains case imply that all the effective interactions are irrelevant; in contrast for a larger number of chains relevant effective interactions are present.

## 1. INTRODUCTION

A quantum system in which disorder-induced localization [1] persists in presence of an interaction is said to be in a Many Body Localized (MBL) phase. While normal systems are expected to approach asymptotically a thermal state (due to interaction "the system acts as his own bath"), this does not happen in a MBL phase [2], [3], [4], a fact with deep theoretical and technological implications. However the interplay of disorder and interaction produces a complex behavior [5], [6] and the existence itself of a MBL phase is quite a non trivial property which is under deep investigation.

In the case of random disorder MBL was established order by order by formal series in any dimension [7], [8], [9], but this does not exclude delocalization due to the possible divergence of the expansions. In one dimension a non-perturbative proof of MBL has been reached [10], [11], but it relies on a still unproven assumption. Numerical evidence of MBL in one dimensional lattices has been obtained in [12], [13], [14].

Also quasi-random disorder in one dimension produces localization in the single particle case, as found in [15] and rigorously proved in [16], [17]. In presence of interaction, a non perturbative proof of ground state localization has been achieved in [18]. Numerical evidence of MBL with quasi random disorder has been found in [19], [20], [21], [22], [23]. One dimensional systems of particles with quasi-random disorder can be realized in cold-atoms experiments [24] and evidence of MBL was claimed.

As a natural step toward higher dimensions we consider an array of interacting fermionic chains with Aubry-Andre' quasi random disorder [15] and coupled by an hopping term. Such model (with spinful fermions) has been realized in cold atoms experiments in [25]. We call  $x = 0, \pm 1, \pm 2, \dots$  the coordinates of the infinite chain and  $y = 0, \dots, L$  the coordinates labeling the chains, and we consider a system of  $N$  spinless fermions with Hamiltonian

$$H_N = \sum_{i=1}^N H_A(x_i) + J_{\perp} \sum_{i=1}^N \nabla_{y_i} + U \sum_{i,j \in 1}^N v(x_i - x_j) \quad (1)$$

where  $v(x_i - x_j) = \delta_{y, x+1}$  and  $H_A$  is the Aubry-Andre'

Hamiltonian

$$H_A(x) = J \nabla_x + \Delta \cos(2\pi(\omega x + \theta)) \quad (2)$$

and  $\nabla_z f(z) = f(z-1) + f(z+1) - 2f(z)$ ; periodic boundary conditions are imposed in  $y$ . The Hamiltonian (1) describes  $L$  fermionic chains, with identical disorder, intra-chain hopping  $J$ , intra-chain interaction  $U$  and inter-chain hopping  $J_{\perp}$ . If  $J_{\perp} = U = 0$  the system reduces to several uncoupled Aubry-Andre' models [15]. The behavior of the eigenfunctions of  $H_A$  (2) depends crucially on the ratio  $\frac{\Delta}{J}$  between the disorder and the hopping; if  $\frac{\Delta}{J} < 2$  the eigenfunctions are quasi-Bloch extended waves while for  $\frac{\Delta}{J} > 2$  are exponentially decaying and Anderson localization occurs [16], [17]. A metal-insulator transition is therefore present varying the strength of the disorder, a feature making quasi-random disorder somewhat similar to random disorder in three dimensions.

The question we address in this paper is if a localized phase persists in the array described by (1), and how the behavior depends on the interplay between the hopping  $J_{\perp}$ , the interaction  $U$  and the number of chains  $L$ . The main theoretical difficulty is that localization is a *non-perturbative* phenomenon; the presence or absence of localization is related to the convergence or divergence of the series, driven by *small divisors* which can produce dangerous factorials. Information is carried by high orders and instability is not signaled by divergences at low orders, as it happens in quantum field theory.

In the single particle case  $H_A$  the small divisors are similar to the ones in the series in Kolmogorov-Arnold-Moser (KAM) theory, whose convergence implies stability in close-to-integrable system, while divergence is related to the onset of chaos. Indeed the eigenfunctions of (2), can be written in series of  $J$  and divisors are of the form  $\phi_x - \phi_y$  with  $x \neq y$ , with  $\phi_x = \Delta \cos(2\pi(\omega x + \theta))$ . In order to get convergence, and as a consequence localization, one needs to assume number theoretical conditions, called Diophantine (see below), to control the size of  $\|(\omega n)\|$  and  $\|(\omega n + 2\theta)\|$ , with  $\|\cdot\|$  the norm on the side 1 torus, see [16], [17]. Such Diophantine conditions are the same assumed in KAM theory. In presence of interaction the small divisors in the expansion for the  $N$ -particle eigenfunctions are much more complex; they are of the form  $E_N(\underline{x}) - E_N(\underline{y})$ , with  $E_N(\underline{x}) = \sum_{i=1}^N \phi_{x_i}$ . No num-

ber theoretical condition is known to control them for  $N > 1$  [26] (for the  $N = 2$  case see [27]).

Even if the construction of all the eigenfunctions of (1) for a generic  $N$  is outside the present analytical possibilities, we can analyze the problem using a different approach, introduced in [18]: we do not consider the expansion for the eigenfunctions but we compute in the thermodynamical limit  $N \rightarrow \infty$  the grand canonical correlations, which at zero temperature becomes the ground state correlations. This approach allows to take advantage of fermionic cancellations and non-perturbative and rigorous information on localization of systems with an infinite number of particles, even if limited to the ground state, can be obtained. The correlations are written as Grassmann integrals which are analyzed via exact fermionic Renormalization Group (RG) methods; one integrates out the degrees of freedom with smaller and smaller energy obtaining a sequence of effective interactions, sum of terms which are all *relevant* in the RG sense, independently from the number of fields. The presence of an infinite number of relevant processes seems to say that an RG approach is hopeless; however by exploiting number theoretical properties of the frequency of the incommensurate disorder it is possible to show that a huge class of effective interactions, called *non resonant*, are indeed *irrelevant*. In contrast with the single chain problem, in which Diophantine conditions are sufficient, here one needs also other conditions called in KAM theory the first and second *Melnikov conditions*. While in absence of interaction the structure of Feynman graphs is rather simple, the presence of interaction  $U \neq 0$  complicates considerably the problem; one has a combination of small divisors and loops, which are absent in non interacting or KAM-like problems. Other dangerous factorials, in addition to the ones produced by small divisors, are produced by combinatorics related to the number of graphs; they are controlled by cancellations due to the fermionic sign cancellations.

A renormalized expansion is obtained in terms of the running coupling constants corresponding to the resonant terms. As usual in RG, the physical properties depend on their flow; if the running coupling constants do not exit from the convergence radius the interacting theory is analytically close to the free one, so that localization persists in presence of interaction. The flow dramatically depends on the number of chains. In the two chain problem there are no relevant effective quartic interactions, the only relevant couplings being quadratic, as in the single chain problem; localization persists in the ground state in presence of interaction.

On the contrary, with an higher number of chains the quartic terms are relevant, and their size increase iterating the RG; therefore a different qualitative behavior is generically expected, unless the many body interaction is vanishing, where localization still persists.

The content of this paper is the following. In §2 we present the main results. In §3 we perform an exact RG analysis and we show the irrelevance of the non resonant

terms. In §4 we identify the relevant and marginal terms and study the corresponding flow, and in §5 we get our main results discussing the convergence of the expansion. Finally in §6 the main conclusions are presented.

## 2. MAIN RESULT

We consider the grand canonical averages  $\langle O \rangle = \sum_N \frac{\text{Tr}_N e^{-\beta(H_N - \mu N)} O}{Z}$ , with  $Z = \sum_N \text{Tr}_N e^{-\beta(H_N - \mu N)}$ ; the thermodynamic limit is taken sending the chain length to infinity keeping the number of chains  $L$  finite. The Fock space Hamiltonian is

$$H = J \sum_{x,y} (a_{x+1,y}^+ a_{x,y}^- + a_{x-1,y}^+ a_{x,y}^-) + \Delta \sum_{x,y} \cos(2\pi\omega x) a_{x,y}^+ a_{x,y}^- + U \sum_{x,y} a_{x,y}^+ a_{x,y}^- a_{x+1,y}^+ a_{x+1,y}^- + J_\perp \sum_{x,y} (a_{x,y+1}^+ a_{x,y}^- + a_{x,y}^+ a_{x,y+1}^-) \quad (3)$$

and we assume for definiteness the phase of the disorder equal to zero. It is convenient to write  $a_{x,y}^\pm = \frac{1}{L} \sum_l e^{\pm i l y} \hat{a}_{x,l}^\pm$ , where  $l = 2\pi \frac{n}{L}$  with  $n = 0, \dots, L-1$  so that the Hamiltonian can be rewritten in the following way

$$H = J \frac{1}{L} \sum_{x,l} (\hat{a}_{x+1,l}^+ \hat{a}_{x,l}^- + \hat{a}_{x-1,l}^+ \hat{a}_{x,l}^-) + \frac{\Delta}{L} \sum_{x,l} \cos(2\pi\omega x) \hat{a}_{x,l}^+ \hat{a}_{x,l}^- + \frac{J_\perp}{L} \sum_{x,l} \cos l (\hat{a}_{x,l}^+ \hat{a}_{x,l}^- + \hat{a}_{x,l}^+ \hat{a}_{x,l}^-) + U \sum_x \frac{1}{L^4} \sum_{l_1, l_2, l_3, l_4} \hat{a}_{x,l_1}^+ \hat{a}_{x,l_2}^- \hat{a}_{x+1,l_3}^+ \hat{a}_{x+1,l_4}^- \delta(l_1 - l_2 + l_3 - l_4) \quad (4)$$

We focus on the 2-point function  $\langle \hat{a}_{\mathbf{x},l}^- \hat{a}_{\mathbf{z},l}^+ \rangle$ , where  $\langle O \rangle = \frac{\text{Tr} e^{-\beta(H - \mu N)} T O}{\text{Tr} e^{-\beta(H - \mu N)}}$ ,  $T$  is the time ordering and  $\hat{a}_{\mathbf{x},l}^\pm = e^{(H - \mu N)x_0} \hat{a}_{x,l}^\pm e^{-(H - \mu N)x_0}$  and  $\mathbf{x} = (x_0, x)$ . In the *molecular limit*  $U = J = 0$  one has (setting  $\Delta = 1$  for definiteness)

$$H_0 - \mu N = \frac{1}{L} \sum_{x,l} (\cos(2\pi\omega x) - \mu_l) \hat{a}_{x,l}^+ \hat{a}_{x,l}^- \quad (5)$$

with

$$\mu_l = \mu + J_\perp \cos l \equiv \cos 2\pi \bar{x}_l \quad (6)$$

so that, calling  $\mu = \cos(2\pi\omega \bar{x})$ , than  $\bar{x}_l = \bar{x} + a \cos l J_\perp$  with  $a^{-1} = \sin 2\pi \bar{x} + O(J_\perp)$ . In this limit the system is uncoupled with an  $l$ -dependent chemical potential for any chain. The ground state occupation number is  $= 1$  for  $\cos(2\pi\omega x) < \mu_l$  and 0 for  $\cos(2\pi\omega x) > \mu_l$ . The 2-point function  $\langle \hat{a}_{\mathbf{x},l}^- \hat{a}_{\mathbf{y},l}^+ \rangle_{U=J=0} \equiv g_l(\mathbf{x}, \mathbf{y})$  is equal to

$$g_l(\mathbf{x}, \mathbf{y}) = \delta_{x,y} \frac{1}{\beta} \sum_{k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})} \hat{g}_l(x, k_0) e^{-ik_0(x_0 - y_0)} \quad (7)$$

with

$$\hat{g}_l(x, k_0) = \frac{\int_0^\beta d\tau e^{i\tau k_0} \frac{e^{-\tau(\cos 2\pi\omega x - \mu_l)}}{1 + e^{-\beta(\cos 2\pi\omega x - \mu_l)}}}{-ik_0 + \cos 2\pi\omega x - \cos 2\pi\omega \bar{x}_l} \quad (8)$$

The 2-point function is perfectly localized in the chain direction (the 2-point function is vanishing if  $x \neq y$ ), but not on the transversal direction in the coordinate space. Assume that  $\bar{x}_l$  is not a point of the lattice, so that the propagator (8) is never singular. As  $\omega$  is an irrational number,  $\omega x$  modulo 1 fills densely the set  $[-1/2, 1/2]$ , and in particular it can be arbitrarily close to  $\pm\omega\bar{x}_l$ . If we set  $x = x' + \rho_l\bar{x}_l$  then for  $(\omega x')_{\text{mod } 1}$  small,  $\rho_l = \pm$

$$\hat{g}_l(x, k_0) \sim \frac{1}{-ik_0 + v_l\rho_l(\omega x')_{\text{mod } 1}} \quad (9)$$

and  $v_l = \sin 2\pi\omega\bar{x}_l$ . The expansion of the 2-point function in terms of  $J, U$  can be represented in terms of Feynman graphs, expressed by product of propagators  $\hat{g}_l(x, k_0)$ ; on each line of the diagram is associated a coordinate  $x$  and the difference of lines coming in or out from the vertex  $J$  is  $\pm 1$ , while from an  $U$  vertex is  $0, \pm 1$ . Note the similarity of (8) with the 2-point function in the free fermion limit  $\Delta = U = J_\perp = 0$ ,  $J = 1$  which in Fourier space is given by  $1/(-ik_0 + \cos k - \mu)$ . If  $k = k' \pm p_F$ ,  $\mu = \cos p_F$  the free fermion propagator is asymptotically given by  $1/(-ik_0 \pm v_F k')$ , which is the well known *Luttinger liquid* propagator.  $p_F$  are called Fermi momenta and by analogy we can call  $\pm\bar{x}_l$  the *Fermi coordinates*.

The expansion in  $J, U$  around the molecular limit is convergent at finite temperature, as the temperature acts as an infrared cut-off, and the main issue is to get the zero temperature limit. We expect that the interaction produces a renormalization of the chemical potential, and it is convenient to fix the renormalized chemical potential to a  $J, U$ -independent value; this corresponds to fix the density of the interacting system. We therefore write

$$\mu_l = \cos 2\pi\omega\bar{x}_l + \nu_l \quad (10)$$

where  $\nu_l$  is a counterterm to be fixed so that the chemical potential of the interacting theory is  $\cos 2\pi\omega\bar{x}_l$ . In order to understand the behavior at high orders one needs to exploit some number theoretical property of  $\omega$ ; in particular, as in the analysis of the Aubry-Andre' model, we assume that the frequency  $\omega$  is a Diophantine number, verifying the property

$$\|\omega x\| \geq C_0|x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (11)$$

with  $\|\cdot\|$  is the norm on the one dimensional torus. Such a property, saying roughly speaking that  $\omega$  is a “good” irrational, is not restrictive as Diophantine numbers have full measure. As an example, the golden ratio  $\omega = \frac{\sqrt{5}+1}{2}$  verifies (11) with  $\tau = 1$  and  $C_0 = \frac{3+\sqrt{5}}{2}$ . The Diophantine condition will ensure that a process involving fermions

living close to  $(\omega\bar{x}_l)$  involves fermions with a huge difference of coordinates.

In addition one has to assume a diophantine condition on the chemical potential (equivalently one can assume a similar condition on  $\theta$ ), namely

$$\|\omega x \pm 2\omega\bar{x}\| \geq C_0|x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (12)$$

with  $\|\cdot\|$  is the norm on the one dimensional torus. Such condition says  $\bar{x}$  is incommensurate with  $\omega$ . In the decoupled case  $J_\perp = 0$  this implies that  $|\hat{g}_l(x, k_0)| \leq C|x|^\tau$ . Our main result is the following.

*If  $U, J_\perp, J$  are small,  $J_\perp \neq 0$  belongs to a set of large relative measure and if  $\omega, \bar{x}$  verify (11) and (12), for a suitable  $\nu_l$ , then:*

*a) if  $L = 2$  for  $\beta \rightarrow \infty$  then for any integer  $N$  and a suitable constant  $C_N$*

$$| \langle \hat{a}_{\mathbf{x},l}^- \hat{a}_{\mathbf{y},l}^+ \rangle | \leq e^{-\xi|x-y|} \frac{C_N |\log \Delta|}{1 + (\Delta|x_0 - y_0|)^N} \quad (13)$$

*with  $\xi = |\log \varepsilon|$  and  $\Delta = (1 + \min(|x|, |y|))^{-\tau}$ .*

*b) If  $L \geq 3$ ,  $U = 0$  then for  $\beta \rightarrow \infty$  (13) holds.*

*c) If  $L \geq 3$ , then (13) holds for  $\beta|U| \leq 1$ , with  $\xi = \max(|\log \varepsilon|, \beta^{-1})$ ,  $\Delta = \max((1 + \min(|x|, |y|))^{-\tau}, \beta^{-1})$*

In the case of two chains (case a) the 2-point function decays at zero temperature exponentially in the direction of the chains, and a very weak decay is present in the imaginary time direction (faster than any power but with rate decreasing increasing  $x, y$ ); this is very similar to what happens in the single chain case and indicates localization of the ground state with or without interaction. In contrast, for a greater number of chains the interaction produces a qualitative difference; in absence of many body interaction, zero temperature exponential decay is found for any number of chains (case b) while in presence of interaction convergence of the expansion holds only up to a finite temperature (case c). The reason is that when  $L \geq 3$  there are extra relevant terms increasing iterating the RG, and this has the effect that convergence holds only for temperatures not too small; as usual, the presence of diverging directions in the RG flow is expected to signal an instability of the system. This provides an explanation of the behavior observed in cold atoms experiments [25], in which absence of localization is found in an array of chains (except when there is no interaction, when localization is found), and localization in the single chain case; moreover, we find localization with two chains in the spinless case, a prediction in principle accessible to future experiments.

### 3. RENORMALIZATION GROUP ANALYSIS

The 2-point function is obtained by the second derivative of the generating function

$$e^{W(\phi)} = \int P(d\psi) e^{V(\psi) + (\psi, \phi)} \quad (14)$$

with

$$\begin{aligned} V = & \frac{1}{L} \sum_l \int d\mathbf{x} J(\psi_{\mathbf{x},l}^+ \psi_{\mathbf{x}+\mathbf{e}_1,l}^- + \psi_{\mathbf{x}+\mathbf{e}_1,l}^+ \psi_{\mathbf{x},l}^-) + \\ & \int d\mathbf{x} \frac{U}{L^4} \sum_l \psi_{\mathbf{x},l_1}^+ \psi_{\mathbf{x},l_2}^- \psi_{\mathbf{x}+\mathbf{e}_1,l_3}^+ \psi_{\mathbf{x}+\mathbf{e}_1,l_4}^- \delta(l_1 - l_2 + l_3 - l_4) \\ & + \frac{1}{L} \sum_l \nu_l \int d\mathbf{x} \psi_{\mathbf{x},l}^+ \psi_{\mathbf{x},l}^- \end{aligned} \quad (15)$$

where  $\psi$  are grassmann variables,  $\phi$  is the external source,  $\int d\mathbf{x} = \int dx_0 \sum_x$ ,  $\mathbf{e}_1 = (0, 1)$  and  $P(d\psi)$  is the fermionic integration with propagator (7)

We introduce a cut-off smooth function  $\chi_\rho(k_0, x)$  which is non vanishing for  $\sqrt{k_0^2 + (v_l(\omega(x - \rho \bar{x}_l)_{\text{mod } 1})^2)} \leq \gamma$ , where  $\rho = \pm 1$  and  $\gamma > 1$  is a suitable constant (to be fixed below); therefore we can write the propagator as

$$\hat{g}_l(\mathbf{x}) = \hat{g}_l^{(u.v.)}(\mathbf{k}) + \sum_{\rho=\pm} \hat{g}_{\rho,l}(\mathbf{k}) \quad (16)$$

where

$$\hat{g}_{\rho,l}(k_0, x) = \frac{\chi_\rho(k_0, x)}{-ik_0 + \cos(2\pi(\omega x)) - \cos(2\pi(\omega \bar{x}_l))} \quad (17)$$

and correspondingly  $\psi_{k_0,x,l} = \psi_{k_0,x,l}^{(u.v.)} + \sum_{\rho=\pm 1} \psi_{\rho;k_0,x,l}$ . This simply says that we are rewriting the fermionic field as sum of two independent fields living close to one of the Fermi points, up to a regular field. We can further decompose

$$\hat{g}_{\rho,l}(k_0, x) = \sum_{h=h_\beta}^0 \hat{g}_{\rho,l}^{(h)}(k_0, x) \quad (18)$$

with  $-h_\beta \sim \log \beta$ ,  $\hat{g}_{\rho,l}^{(h)}(k_0, x)$  similar to  $\hat{g}_{\rho,l}(k_0, x)$  with  $\chi$  replaced by  $f_h$  with  $f_h(k_0, \omega x')$  non vanishing in a region  $\sqrt{k_0^2 + (v_l(\omega x')_{\text{mod } 1})^2} \sim \gamma^h$  with  $x = x' + \rho \bar{x}_l$ .

After the integration of the fields  $\psi^{(u.v.)}, \psi^{(0)}, \dots, \psi^{(h+1)}$  the generating function has the form

$$e^{W(\phi)} = \int P(d\psi^{\leq h}) e^{V^{(h)}(\psi) + B^{(h)}(\psi, \phi)} \quad (19)$$

where  $P(d\psi^{\leq h})$  has propagator  $g_{\rho,l}^{(\leq h)} = \sum_{k=-\infty}^h g_{\rho,l}^{(k)}$  and  $V^{(h)}(\psi)$  is given by sum of terms

$$\begin{aligned} & \sum_{x'_1} \int dx_{0,1} \dots \int dx_{0,m} \frac{1}{L^m} \sum_{l_1, \dots, l_m} W_{m,l}^{(h)}(x'_1, \underline{x},) \\ & \delta(\sum_i \varepsilon_i l_i) \psi_{\rho_1; x_{0,1}, x'_1, l_1}^{\varepsilon_1(\leq h)} \dots \psi_{\rho_m; x_{0,m}, x'_m, l_m}^{\varepsilon_m(\leq h)} \end{aligned} \quad (20)$$

where the kronecker deltas in the propagators imply that a single sum over  $x$  is present; the kernels  $W_m^{(h)}$  are sum of Feynman diagrams obtained connecting vertices  $J, U$  or  $\nu$  with propagators  $g^{(k)}$  with  $k > h$ . Similarly  $B^{(h)}$  is given by a similar expression with the only difference that some of the external lines are associated to  $\phi$  fields. The scaling dimension of the theory can be obtained by the bounds

$$\int dx_0 |g_\rho^{(h)}(x_0, x)| \leq C \gamma^{-h} \quad |g_\rho^{(h)}(x_0, x)| \leq C \quad (21)$$

The persistence or not of localization is related to the presence or lack of convergence, that is the behavior at high orders; we need therefore to remind some basic tool of renormalization theory, which are crucial to avoid the well known problem of "overlapping divergences". Given a Feynman graph, one considers a maximally connected subset of lines corresponding to propagators with scale  $h \geq h_v$  with at least a scale  $h_v$ , and we call it *cluster v* (for more details, see [28]); the external lines have scale smaller than  $h_v$ . Therefore to each Feynman graph is associated a hierarchy of clusters; inside each cluster  $v$  there are  $S_v$  maximal clusters, that is clusters contained only in the cluster  $v$  and not in any smaller one, or trivial clusters given by a single vertex. The clusters therefore identify the subdiagrams which one needs to renormalize, as the ones containing propagators living at energy scales greater than the ones outside them.

Each of such  $S_v$  clusters are connected by a tree of propagators with scale  $h_v$ ; by integrating the propagators over time and using (21) we get that each graph of order  $n$  contributing to  $W_m^{(h)}$  is bounded at fixed scale by, if  $\varepsilon = \max(|J|, |U|)$

$$C^n \varepsilon^n \prod_v \gamma^{-h_v(S_v-1)} \quad (22)$$

where  $v$  are the clusters (not end-points) and  $h_v \leq 0$ . From the above estimate we see that the scaling dimension of any contribution to the effective potential has the same positive scaling dimension (independently from the number of fields)

$$D = 1 \quad (23)$$

In other words all the effective interactions are *relevant* in the RG sense and the theory is *non-renormalizable*; indeed to the effective potential graphs with all the assignments of scales contribute and from (22) the sum over scales gives an infinite result. However, it turns out, as a consequence of number theoretical properties of the quasi random disorder, that a huge class of terms are indeed *irrelevant*. In a large relative measure set of  $J_\perp$  one has

$$||\omega x \pm 2\omega \bar{x}_l|| \geq C_0 |x|^{-\tau'} \quad \forall x \in \mathbb{Z}/\{0\} \quad (24)$$

and

$$||\omega x \pm \omega \bar{x}_l \pm \omega \bar{x}_{l'}|| \geq C_0 |x|^{-\tau'} \quad \forall x \in \mathbb{Z}/\{0\} \quad (25)$$

Conditions (24) and (25) are known in KAM theory as the first and second *Melnikov conditions*. The first condition is used to bound the propagator; using that  $||\omega x'|| = ||\omega x - \rho\omega\bar{x}_l|| = ||\omega 2x - 2\rho\omega\bar{x}_l||$  for  $||\omega x'||$  small then  $|\hat{g}^{(h)}(k_0, x)| \leq C|x|^\tau$ . The second condition is used to show the irrelevance of a number of terms in the effective potential. Let us consider a contribution to the effective potential (20) with external lines  $\psi_{\rho_1; x_0, 1, x'_1, l_1}^{\varepsilon_1(\leq h)} \dots \psi_{\rho_m; x_0, m, x'_m, l_m}^{\varepsilon_m(\leq h)}$ . By construction the coordinates of the external fields are such that  $(\omega x')_{\text{mod } 1} \leq \gamma^h$ . Note that in each graph there is a tree of propagators connecting all the vertices and external lines; each propagator carries a coordinate  $x$  and vertices connect lines with coordinates differing at most of  $\pm 1$ ; more exactly, if  $x_i, x_j$  are the coordinates of two external lines

$$x_i - x_j = x'_i + \rho_i \bar{x}_{l_i} - x'_j - \rho_j \bar{x}_{l_j} = \sum_{\alpha}^* \delta_{\alpha} \quad (26)$$

where the sum is over the vertices in the path of the tree connecting  $i$  and  $j$  and  $\delta_{\alpha} = (0, 1, -1)$  is associated to the line connected to the vertex  $\alpha$ . When  $U = 0$  then necessarily  $l_j = l_i$ . It is natural to distinguish among the terms contributing to the effective potential between *resonant terms* and the *non resonant terms*. The first are the contributions in (20) in which the coordinate  $x'$  of the external fields are equal  $\psi_{\rho_1; x_0, 1, x'_1, l_1}^{\varepsilon_1(\leq h)} \dots \psi_{\rho_m; x_0, m, x'_1, l_m}^{\varepsilon_m(\leq h)}$ , that is for any  $i, j$

$$x'_i = x'_j \quad (27)$$

The non resonant terms are the ones such that, for some  $i, j$ ,  $x'_i \neq x'_j$  so that from (26) and the second Melnikov condition (25)

$$2\gamma^h \geq ||(\omega x'_i)|| + ||(\omega x'_j)|| \geq ||\omega(x'_i - x'_j)|| = (28)$$

$$||\omega(\rho_i \bar{x}_{l_i} - \rho_j \bar{x}_{l_j}) + \omega \sum_{\alpha}^* \delta_{\alpha}|| \geq \frac{C_0}{|\sum_{\alpha}^* \delta_{\alpha}|^{\tau'}}$$

so that  $|\sum_{\alpha}^* \delta_{\alpha}| \geq \tilde{C}\gamma^{-h/\tau}$ .

One can then use the high power of  $J, U$  to get a gain factor making irrelevant the non resonant contributions to the effective potential. Writing  $\varepsilon = \max(|J|, |U|)$ ,  $\varepsilon = \prod_{h=-\infty}^0 \varepsilon^{2^{h-1}}$ , we can associate a factor  $\varepsilon^{2^{h_v-1}}$  for each end-point enclosed in the cluster  $v$ ; as  $|\sum_{\alpha}^* \delta_{\alpha}|$  is surely smaller than the number of vertices in the cluster  $v$  and choosing  $\gamma^{\frac{1}{\tau}}/2 > 1$ , we can associate to each non resonant contribution a factor  $\varepsilon^{2^{h_v-1}|\sum_{\alpha}^* \delta_{\alpha}|} \leq \varepsilon^{C2^{h_v}\gamma^{-h/\tau}} \leq \gamma^{4h}$  for  $\varepsilon$  small; therefore

$$\varepsilon^{\frac{n}{2}} \leq \prod_v \varepsilon^{C2^{h_v}\gamma^{-h_v/\tau} S_v^{NR}} \leq \prod_v \gamma^{4h_v S_v^{NR}} \quad (29)$$

where  $S_v^{NR}$  is the number of non resonant clusters in  $v$ ; this means that to each non resonant term is associated at least a factor  $\gamma^{4h_v}$  which is sufficient to make its scaling dimension negative.

It remains to prove that (24) and (25) are true in a large relative measure set of values, that is if  $|J_{\perp}| \leq \varepsilon_0$ , in a set of whose complement has measure  $O(C_L \varepsilon_0^{1+\alpha})$ ,  $\alpha \geq 0$ ,  $C_L$  an  $L$ -dependent constant. Indeed if (24) is true then, if  $\cos l \neq 0$

$$C_0|x|^{-\tau} \leq ||\omega x \pm 2\bar{x}|| \leq ||\omega x \pm 2(\bar{x} + aJ_{\perp} \cos l)|| + 2|aJ_{\perp} \cos l| \leq C_0|x|^{-\tau'} + C|\varepsilon_0 \cos la| \quad (30)$$

so that if  $\tau' > \tau + 1$ ,  $C_0/2|x|^{-\tau} \leq C_0|x|^{-\tau}(1 - |x|^{\tau-\tau'}) \leq C|a\varepsilon_0 \cos l|$  for  $|x| \geq 2$  and  $|x| \geq (2C\varepsilon_0 \cos la/C_0)^{\frac{1}{\tau-\tau'}} = \mathcal{N}_0$ . The set  $I$  of  $J_{\perp}$  not verifying (24) is defined by the condition, for  $-1 \leq s \leq 1$

$$f(s) = \omega x \pm 2(\bar{x} + J_{\perp}(s) \cos la) = sC_0|x|^{-\tau'} \quad (31)$$

and  $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial J_{\perp}} \frac{\partial J_{\perp}}{\partial s} = C_0|x|^{-\tau'}$  so that the measure of the region in which (24) is not true is

$$\begin{aligned} \int_I dJ_{\perp} &= \sum_l \sum_{n \geq \mathcal{N}_0} \int_{-1}^1 \left| \frac{dJ_{\perp}}{ds} \right| ds = \\ &\leq \sum_{l, \cos l \neq 0} \frac{C}{\cos l} \sum_{x \geq \mathcal{N}_0} |x|^{-\tau'} \leq C_L |\varepsilon_0|^{\frac{\tau'-1}{\tau}} \end{aligned} \quad (32)$$

and choosing  $\frac{\tau'-1}{\tau} > 1$ , that is  $\tau' > \tau + 1$  we have that for  $|J_{\perp}| \leq \varepsilon_0$  the relative measure of the excluded  $J_{\perp}$  is  $O(C_L \varepsilon_0^{\tau'-\tau-1})$ , hence vanishing if  $\varepsilon_0 \rightarrow 0$ . A similar procedure can be repeated for the second Melnikov condition; if  $\cos l_i \pm \cos l_j \neq 0$  then

$$C_0|x|^{-\tau} \leq ||\omega x \pm (\bar{x} + a \cos l_i J_{\perp}) \pm (\bar{x} + a \cos l_j J_{\perp})|| + |J_{\perp} a (\cos l_i \pm \cos l_j)| \leq C_0|x|^{-\tau'} + C|\varepsilon_0 (\cos l_i \pm \cos l_j) a|$$

from which  $|x| \geq (2C\varepsilon_0 |(\cos l_i \pm \cos l_j) a|/C_0)^{\frac{1}{\tau-\tau'}}$ ; one then proceeds as above with  $|\cos l_i \pm \cos l_j|$  replacing  $|\cos l_i|$ .

#### 4. THE RESONANT TERMS

We have seen in the preceding section that the non resonant terms are irrelevant. We have then to construct a renormalized expansion for the 2-point function, extracting, at each RG iteration, the marginal and relevant part of the resonant terms. In this way the two-point function is written as an expansion in a set of running coupling constants, which is convergent if such constants remain small at each scale; convergence at the end implies localization in the ground state at a non-perturbative level, as it means that the interacting theory is analytically close to the non interacting one, which is localized.

We focus now on some properties of the resonant terms. Note that  $x_i - x_j = x'_i - x'_j + \rho_i \bar{x}_{l_i} - \rho_j \bar{x}_{l_j} \in \mathbb{Z}$  so that in the resonances  $\rho_i \bar{x}_{l_i} - \rho_j \bar{x}_{l_j} \in \mathbb{Z}$ . This says that, up to a zero measure set of  $J_{\perp}$ ,  $\rho_i \bar{x}_{l_i} - \rho_j \bar{x}_{l_j} = 0$  as  $(\cos l_i - \cos l_j) a J_{\perp}$  or  $2\bar{x} + (\cos l_i + \cos l_j) a J_{\perp}$  can be a non vanishing integer only in a zero measure set (by the

diophantine condition  $2\bar{x}$  cannot be integer). In addition in a resonant terms necessarily all the fields have the same  $\rho$

$$\rho_i = \rho_j \quad (33)$$

as if  $\rho_i = -\rho_j$  one get  $2\bar{x} + (\cos l_i + \cos l_j)aJ_\perp = 0$  which cannot be vanishing for small  $J_\perp$ . Finally if  $\cos l_i \neq \cos l_j$  then necessarily in the resonance  $l_j = l_i$ , as the condition becomes  $(\cos l_i - \cos l_j)aJ_\perp = 0$ . The above properties imply that the resonances with a number of fields  $\geq 4$  have the following structure

$$\prod_i \psi_{\rho; x', x_{0,i}, l_i}^{\varepsilon_i} \quad \cos l_j = \cos l_i \quad (34)$$

If  $L = 2$ , that is the array is only composed by two chains then  $l = (0, \pi)$ ,  $\bar{x}_1 = \bar{x} + J_\perp/2$  and  $\bar{x}_2 = \bar{x} - J_\perp/2$  so that the resonant terms have the same  $\rho, l$  index; this has the effect that the monomials with  $\geq 4$  fields and the same coordinates are vanishing. In the resonances with a number of fields greater than two there are at least two couples of the form  $\psi_{\rho; x', x_{0,1}, l}^\varepsilon \psi_{\rho; x', x_{0,2}, l}^\varepsilon$  which can be rewritten as

$$\psi_{\rho; x', x_{0,1}, l}^\varepsilon \psi_{\rho; x', x_{0,2}, l}^\varepsilon = \psi_{\rho; x', x_{0,1}, l}^\varepsilon (\psi_{\rho; x', x_{0,2}, l}^\varepsilon - \psi_{\rho; x', x_{0,1}, l}^\varepsilon) \quad (35)$$

and

$$\psi_{\rho; x', x_{0,2}, l}^\varepsilon - \psi_{\rho; x', x_{0,1}, l}^\varepsilon = (x_{0,2} - x_{0,1}) \int_0^1 dt \partial \psi_{\rho; x', x_{0,1}(t), l}^\varepsilon \quad (36)$$

with  $x_0(t) = x_{0,1} + t(x_{0,2} - x_{0,1})$ . The derivative produces an extra  $\gamma^{h_{v'}}$ , if  $v'$  is the cluster enclosing  $v$ , and the factor  $(x_{0,2} - x_{0,1})$  an extra  $\gamma^{-h_v}$ ; as there are at least two of such monomials one gets at least a factor  $\gamma^{2(h_{v'} - h_v)}$ . Remembering that the scaling dimension is  $D = 1$ , this means that all the resonances with more than two fields are *irrelevant* if  $L = 2$ .

If  $L \geq 3$  the situation is different; there are couple of indices  $l, l'$  such that  $x_l = x_{l'}$ ; quartic terms involving such couple of indices and the same  $x_{0,i}$  are not vanishing so that there are quartic relevant terms. For instance in the three chains problem  $L = 3$  one has  $l = 2\pi/3, 4\pi/3, 6\pi/3$  and  $\bar{x}_1 = \bar{x} - J_\perp/2, \bar{x}_2 = \bar{x} - J_\perp/2, \bar{x}_3 = \bar{x}$ ; the local part of the quartic terms (the part with identical coordinates)  $\psi_{\rho; x', 1}^+ \psi_{\rho; x', 1}^- \psi_{\rho; x', 2}^+ \psi_{\rho; x', 2}^-$  is non vanishing; the quartic terms are indeed relevant while resonant terms with a number greater than 6 are irrelevant. The number of couples  $i, j$  with  $\cos l_i = \cos l_j$ , and the corresponding quartic terms, increases with  $L$ ; for instance for  $L = 8$  one has  $l = \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4, 2\pi$  with  $\cos l = \sqrt{2}/2, 0, -\sqrt{2}/2, -1, -\sqrt{2}/2, 0, \sqrt{2}/2, 1$ , so that the non vanishing local quartic terms are  $\psi_{\rho; x', 1}^+ \psi_{\rho; x', 1}^- \psi_{\rho; x', 7}^+ \psi_{\rho; x', 7}^-$ ,  $\psi_{\rho; x', 2}^+ \psi_{\rho; x', 2}^- \psi_{\rho; x', 6}^+ \psi_{\rho; x', 6}^-$ ,  $\psi_{\rho; x', 3}^+ \psi_{\rho; x', 3}^- \psi_{\rho; x', 5}^+ \psi_{\rho; x', 5}^-$ . As there are at most couples of fields with the same  $\bar{x}_l$  and different  $l$ , the terms with a number  $\geq 6$  of fields are irrelevant, as there are at least 4 fields with the same  $l$ .

In order to get a convergent expansion, one has to extract the relevant part from the resonant terms. If  $V_{res}^h = \sum_m V_{m, res}^h$  where  $V_m^h$  are the monomials with  $m$  fields, then we define a localization operation  $V_2^h = \mathcal{L}V_2^h + \mathcal{R}V_2^h$  with  $\mathcal{R} = 1 - \mathcal{L}$  and  $\mathcal{L}$  acts on the kernels of  $V_2^h$  in the following way

$$\mathcal{L}\widehat{W}_2^h(k_0, x') = \widehat{W}_2^h(0, 0) + k_0 \partial_0 \widehat{W}_2^h(0, 0) + (\omega x') \tilde{\partial} W_2^h(0, 0) \quad (37)$$

where  $\tilde{\partial} \widehat{W}_2^h(k_0, x') = \frac{\widehat{W}_2^h(k_0, x') - \widehat{W}_2^h(k_0, 0)}{(\omega x')}$ . The action of  $\mathcal{R} = 1 - \mathcal{L}$  produces a gain  $\gamma^{2(h_{v'} - h_v)}$ , using also that  $(\omega x')^2 \sim \gamma^{2h_{v'}}$ , if  $v'$  is the smallest cluster enclosing  $v$ , for the compact support properties of the lines external to the cluster  $v$ , while  $\tilde{\partial}^2 \widehat{W}_2^h(k_0, x')$  has an extra  $\gamma^{-2h_v}$ ;  $\mathcal{R}V_2^h$  is therefore irrelevant. The local part of the effective potential has then the form

$$\mathcal{L}V_{res, 2}^h = \sum_x \frac{1}{L} \sum_{\rho, l} \int dx_0 (\nu_{h, l} \gamma^h \psi_{\rho; x', l}^+ \psi_{\rho; x', l}^+ + z_{h, l, \rho} \psi_{\rho; x', l}^+ \partial_0 \psi_{\rho; x', l}^+ + \alpha_{h, l, \rho} (\omega x') \psi_{\rho; x', l}^+ \psi_{\rho; x', l}^+) \quad (38)$$

Regarding terms with a number of fields  $\geq 6$ ,  $\mathcal{L}V_{res, m}^h = 0$  for  $m \geq 6$ , as the local part and its first derivative are vanishing. Finally if  $L = 1, 2$  then

$$\mathcal{L}V_{res, 4}^h = 0 \quad (39)$$

while for  $L \geq 3$  then

$$\mathcal{L}V_{res, 4} = G + \frac{1}{L^3} \sum_{i, j; \bar{x}_i = \bar{x}_j} \sum_{\rho} \lambda_{h, i, j, \rho} \gamma^h \int d\mathbf{x} \psi_{\rho; \mathbf{x}', i}^+ \psi_{\rho; \mathbf{x}', i}^- \psi_{\rho; \mathbf{x}', j}^+ \psi_{\rho; \mathbf{x}', j}^- \quad (40)$$

where in  $G$  are included marginal terms, that is quartic local terms with at least a field  $\partial\psi$  (the corresponding coupling are called  $\tilde{\lambda}_{h, i, j, \rho}$ ) and the sum  $\sum_{i, j; \bar{x}_i = \bar{x}_j}$  is over the fields with the same  $\bar{x}_l$ .

## 5. CONVERGENCE OF THE RENORMALIZED EXPANSION

The integration is done separating at each integration step the relevant and the irrelevant part of the effective integration, writing

$$\int P(d\psi^{\leq h}) e^{\mathcal{L}V^{(h)}(\psi^{\leq h}) + \mathcal{R}V^{(h)}(\psi^{\leq h})} \quad (41)$$

with  $\mathcal{R} = 1 - \mathcal{L}$  and  $\mathcal{L}$  is the localization operator defined above; this allows to get an expansion in terms of running coupling constants  $\tilde{v}_h = (\lambda_{h, l, l', \rho}, \tilde{\lambda}_{h, l, l', \rho}, \nu_{h, l}, \alpha_{h, l}, z_{h, l})$ . If  $v_0$  is the largest cluster,  $v$  are the clusters (without vertices),  $\bar{v}$  the vertices and  $R$  or  $NR$  the resonant clusters or vertices and  $v'$  is the first cluster enclosing  $v$ , then

$$\prod_v \gamma^{-h_v S_v} = \prod_{v \neq v_0} \gamma^{-h_{v'}} \prod_{\bar{v}} \gamma^{-h_{\bar{v}'}}$$

and  $\prod_v \gamma^{h_v} = \gamma^{h_{v_0}} \prod_{v \neq v_0} \gamma^{h_v}$  so that (22) can be rewritten as

$$\varepsilon^n \gamma^{h_{v_0}} \prod_{v \neq v_0} \gamma^{-(h_{v'} - h_v)} \prod_{\bar{v}} \gamma^{-h_{\bar{v}'}} \quad (42)$$

Using (29) we get that the kernel  $W_m^{(h)}$  in the renormalized expansion are bounded by, if  $|v_h| \leq \varepsilon$

$$\varepsilon^{n/2} \prod_v \gamma^{-h_v(S_v-1)} \left[ \prod_{v \in R} \gamma^{2(h_{v'} - h_v)} \right] \left[ \prod_{\bar{v} \in R} \gamma^{h_{\bar{v}'}} \right] \left[ \prod_v \gamma^{4S_v^{NR} h_v} \right] \quad (43)$$

where the factor  $\left[ \prod_{v \in R} \gamma^{2(h_{v'} - h_v)} \right]$  is, as explained in the previous section, due to the action of  $\mathcal{R}$  or to (35), (36). Therefore (43) can be written as

$$C^m \varepsilon^{n/2} \gamma^{h_{v_0}} \left[ \prod_v \gamma^{(h_{v'} - h_v)} \right] \left[ \prod_{\bar{v} \in NR} \gamma^{h_{\bar{v}'}} \right] \quad (44)$$

As  $h_{v'} - h_v \leq 0$  it is possible over the scales  $h_v$  obtaining a bound from which convergence follows provided that  $\varepsilon$  is not too large. Note that the above bound is valid for the sum of all Feynman graph of order  $n$ , by using determinant bounds for fermionic expectations, see [28], for details. The renormalized expansion has a finite radius of convergence in terms of the running coupling constants; if they remain, for any  $h$ , inside the convergence radius then localization is found. We have then to analyze the flow of the effective couplings, and the result is dramatically different in the  $L = 2$  and  $L \geq 3$ .

In the case  $L = 2$  there are no quartic terms in the effective potential, that is  $\lambda_h = \tilde{\lambda}_h = 0$ ; the only effective couplings are quadratic and the flow equations are

$$\nu_{h-1,l} = \gamma \nu_h + \beta_{h,l}^\nu \quad \alpha_{h-1,l} = \alpha_h + \beta_{h,l}^\alpha \quad (45)$$

and  $z_{h-1} = z_h + \beta_{h,l}^z$ , where  $\beta_h^\nu, \beta_h^\alpha, \beta_h^z$  are the beta functions; they are given by sum of terms with at least an irrelevant term, as terms containing only marginal terms (quadratic in the fields) are chain graphs giving a vanishing contribution to be beta function by the compact support properties of the propagator. Therefore, by (44), the beta function is asymptotically vanishing  $\beta_h^\nu, \beta_h^\alpha, \beta_h^z = O(\gamma^h)$ .  $\nu_{h,l}$  is a relevant coupling but his flow can be controlled by choosing properly the counterterms  $\nu_l$ ; indeed we can write  $\nu_{h-1,l} = \gamma^{-h}(\nu_l + \sum_{k=h}^0 \gamma^k \beta_{k,l}^\nu)$  and choosing  $\nu_l = -\sum_{k=-\infty}^0 \gamma^k \beta_{k,l}^\nu$  we get that  $\nu_{h,l} = O(\gamma^h \varepsilon)$ .

Moreover  $\alpha_{h-1,l} = \sum_{k=h}^0 \beta_{k,l}^\alpha = O(\varepsilon)$  and similarly  $\alpha_{h-1,l} = O(\varepsilon)$ . Therefore if  $J, U$  are sufficiently small we have that the running coupling constants are small and the series are convergent. Similarly if  $L \geq 3$  and  $U = 0$  there are only quadratic couplings and we can proceed in the same way.

In the case  $L \geq 3$  there are however quartic relevant and marginal couplings, that is

$$\lambda_{h-1,l,l',\rho} = \gamma \lambda_{h,l,l',\rho} + \beta_{h,l,l',\rho}^\lambda \quad (46)$$

Convergence is achieved at finite temperatures, that is for  $\gamma^{-h\beta} U$  or  $\beta U$  of order 1, and at lower temperatures one expects generically an unbounded flow.

An estimate for the 2-point function follows easily from the expansion for the effective potential; if the external coordinates are  $x$  and  $y$  then there are at least  $|x - y| \varepsilon$  factors, and this implies exponential decay in the direction of the chains. By the first Melnikov condition the smallest scale of the contribution at order  $n$  verifies

$$\gamma^{-\bar{h}} \leq C(1 + \min\{|x|, |y|\})^\tau (1 + \frac{n}{1 + \min\{|x|, |y|\}})^\tau \quad (47)$$

from which (13) follows.

## 6. CONCLUSIONS

We have considered an array of interacting chains with quasi-random disorder. The RG analysis provides an explanation of cold-atoms experiments [24], [25], in which it is found that localization is present in the single chain case, while is absent when several chains are considered. In the first case number theoretical properties, combined with cancellations due to Pauli principle, ensure that all the effective interactions are irrelevant, even if dimensionally relevant. On the contrary, in the second case there are non vanishing relevant interactions, whose number increases with the number of chains; as usual, the presence of diverging directions in the RG flow is expected to signal an instability of the system. In addition, we have shown that localization in the ground state is present with two chains, if the fermions are spinless and in presence of interaction, a prediction in principle accessible at an experimental verification.

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